6.2 Linear and Almost Linear Systems (continued)

last time: $x' = F(x,y)$
$y' = G(x,y)$

is almost / locally linear if near a critical pt (x_0, y_0)

the system acts like $\vec{x}' = A\vec{x} + \vec{g}(\vec{x})$

such that $\displaystyle\lim_{(x,y)\to(x_0,y_0)} \frac{|\vec{g}|}{|\vec{x}|} = 0$

for each (x_0, y_0), there is a different A matrix near (x_0, y_0)

example from last time: $x' = -x + xy$
$y' = -2y + 8xy$
cp: $(0,0)$, $(\frac{1}{4}, 1)$ ← look at this one

define $u = x - \frac{1}{4}$ $\quad v = y - 1$ $\qquad x = u + \frac{1}{4}$ $\qquad y = v + 1$

$u' = x'$ $\qquad v' = y'$

system in terms of u, v: $\quad u' = -(u+\frac{1}{4}) + (u+\frac{1}{4})(v+1) = \frac{1}{4}v + uv$

$v' = \cdots = 8u + 8uv$

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \frac{1}{4} \\ 8 & 0 \end{bmatrix}}_{A \text{ near } (\frac{1}{4}, 1)} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} uv \\ 8uv \end{bmatrix}$$

a more efficient way: linearization of the system near (x_0, y_0)

Taylor series

$x' = F(x,y)$

$y' = G(x,y)$ $\qquad$ cp: (x_0, y_0)

higher order (not important)

$$F(x,y) = \cancel{F(x_0,y_0)}^{\,0} + \frac{\partial F}{\partial x}(x_0,y_0)\overbrace{(x-x_0)}^{u} + \frac{\partial F}{\partial y}(x_0,y_0)\overbrace{(y-y_0)}^{v} + \cdots$$

$$G(x,y) = \cancel{G(x_0,y_0)} + \frac{\partial G}{\partial x}(x_0,y_0)\underbrace{(x-x_0)}_{u} + \frac{\partial G}{\partial y}(x_0,y_0)\underbrace{(y-y_0)}_{v} + \cdots$$

def. of crit. pt

system becomes

$$u' = F_x(x_0,y_0)\,u + F_y(x_0,y_0)\,v$$

$$v' = G_x(x_0,y_0)\,u + G_y(x_0,y_0)\,v$$

$$\boxed{\begin{bmatrix} u' \\ v' \end{bmatrix} = \underbrace{\begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix}}_{\text{Jacobian matrix (function of } x,\, y)} \begin{bmatrix} u \\ v \end{bmatrix}}$$

linearized system near (x_0, y_0)

<u>example</u>

$x' = x^2 + y^2 - 6 = F$

$y' = x^2 - y = G$

cp: $x^2 + y^2 - 6 = 0$ $\quad$ $x^2 - y = 0 \rightarrow y = x^2$

$y + y^2 - 6 = 0$

$y^2 + y - 6 = 0$

$(y+3)(y-2) = 0$ $\quad$ $y = -3, \quad y = 2$

$x^2 = y \rightarrow x = \pm\sqrt{y}$ $\quad$ cp: $(\sqrt{2}, 2), (-\sqrt{2}, 2)$

linearize using the Jacobian matrix

$$J(x,y) = \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix} = \begin{bmatrix} 2x & 2y \\ 2x & -1 \end{bmatrix}$$

$$J(\sqrt{2},2) = \begin{bmatrix} 2\sqrt{2} & 4 \\ 2\sqrt{2} & -1 \end{bmatrix} \leftarrow \vec{x}\,' = \begin{bmatrix} 2\sqrt{2} & 4 \\ 2\sqrt{2} & -1 \end{bmatrix}\vec{x}$$

$\lambda \approx 4.8, -3$

saddle pt, unstable

not sensitive to perturbation

linearization does not "lie"

about this being a saddle pt

$\cancel{J(\pm\sqrt{2},}$

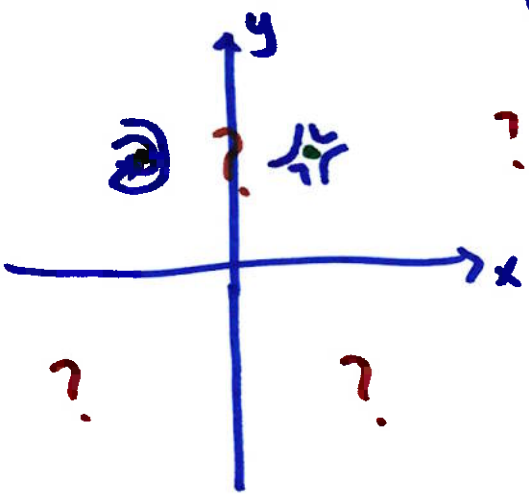
$$J(-\sqrt{2},2) = \begin{bmatrix} -2\sqrt{2} & 4 \\ -2\sqrt{2} & -1 \end{bmatrix}$$

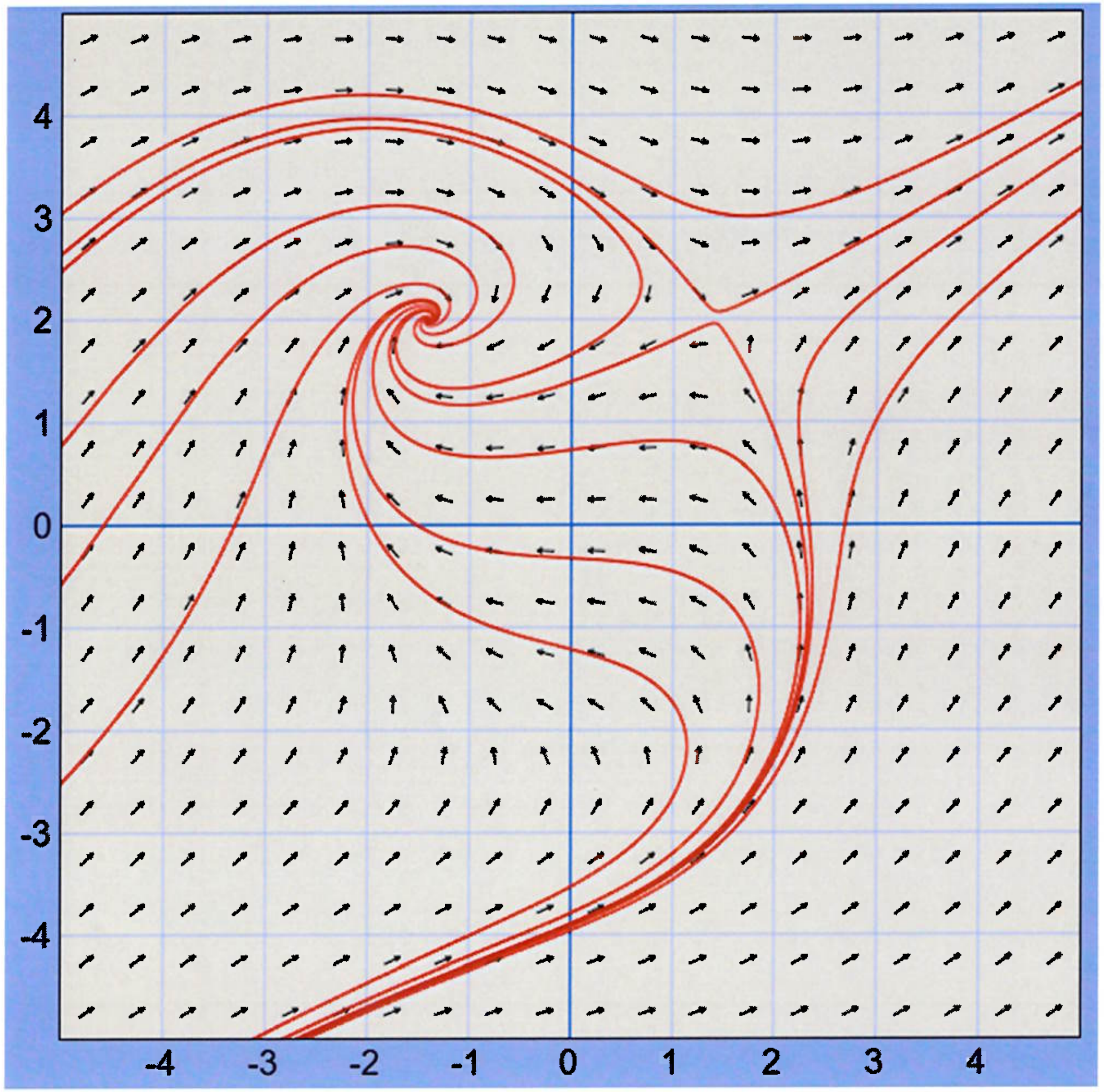
$\lambda \approx -2 \pm 3.2i$

spiral sink

asympt. stable

not sensitive to perturbation





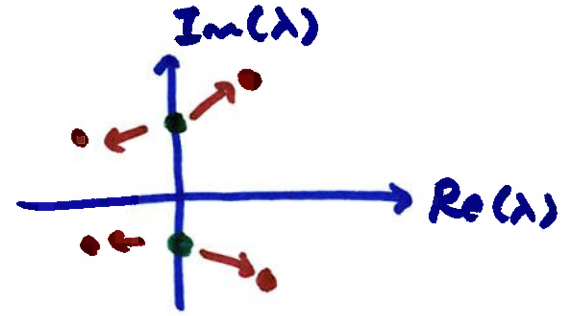
4
3
2
1
0
-1
-2
-3
-4
-4 -3 -2 -1 0 1 2 3 4

example

$x' = 2xy = F$

$y' = 1 - x^2 + y^2 = G$

cp: $(1,0)$, $(-1,0)$



Jacobian: $J(x,y) = \begin{bmatrix} 2y & 2x \\ -2x & 2y \end{bmatrix}$

$J(1,0) = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$ $\quad \lambda = \pm 2i$

linearization says center, stable

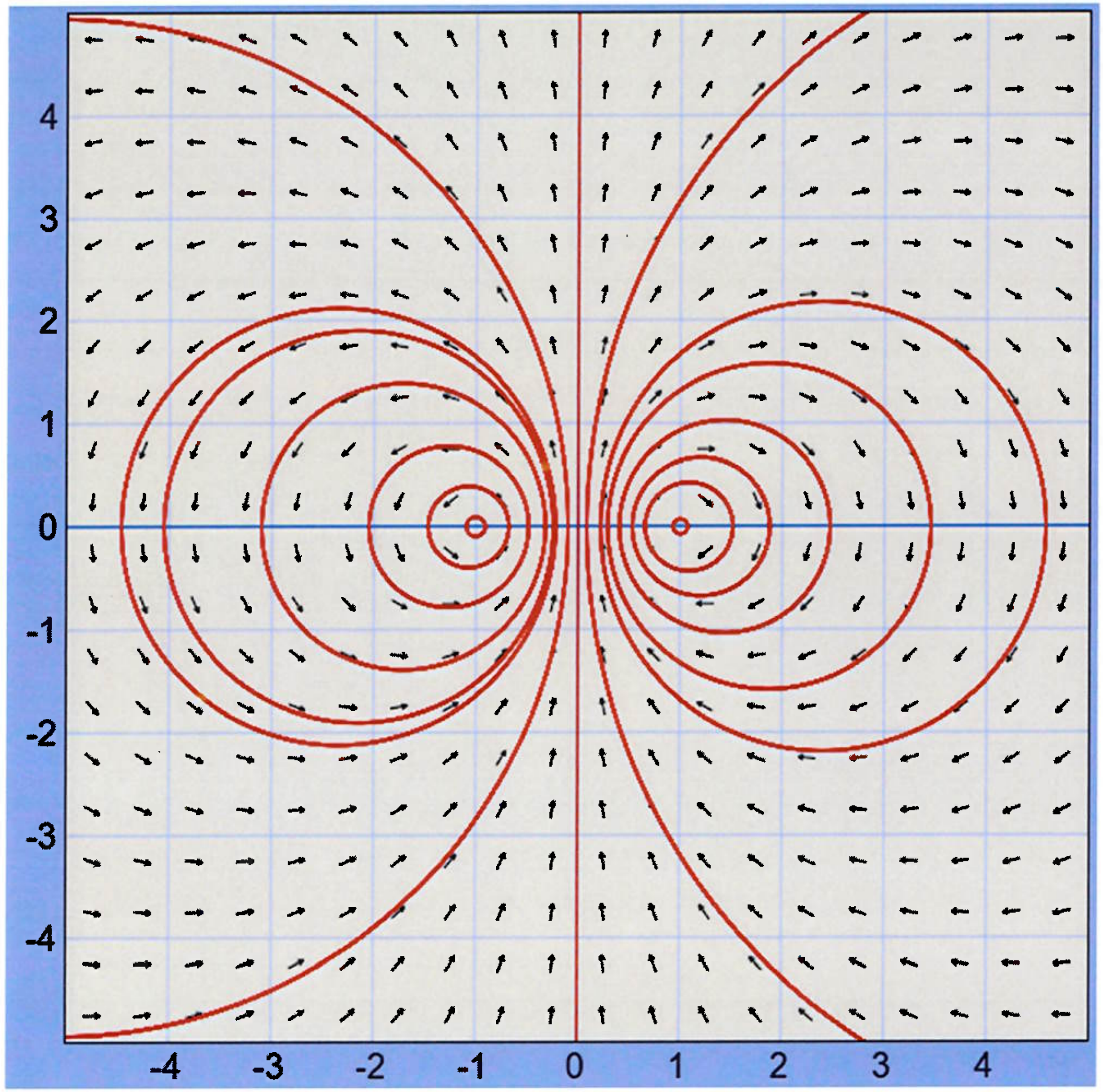
but true behavior of the nonlinear system may or may not be a center

$J(-1,0) = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$ $\quad \lambda = \pm 2i$ same story

to determine true behavior, we can graph the nonlinear phase portrait or solve the system

(this ~~eq~~ one can be solved using sub $u = y^2/x$)

linearization doesn't "lie" but not always the case in general



<u>example</u>

$$x' = -3y + ay(x^2+y^2)$$

$$y' = 3x + ay(x^2+y^2)$$

a is some constant

cp: $(0,0)$

$$J(x,y) = \begin{bmatrix} 2axy & -3+ax^2+3ay^2 \\ 3+2axy & ax^2+3ay^2 \end{bmatrix}$$

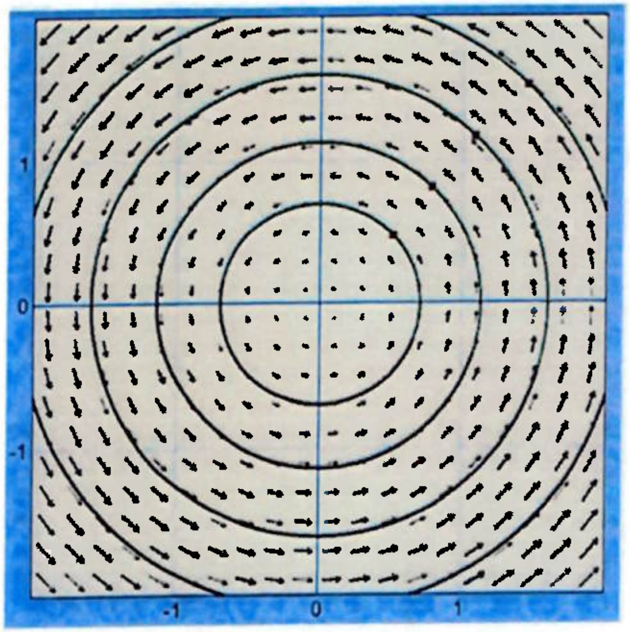
$$J(0,0) = \begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix}$$

linearization lost a completely

linearization cannot capture effect of a

linearization: center, stable

cannot trust : may or may not still be a center



Linearized
(stable center)

Nonlinear a=1
(unstable
spiral point)

Nonlinear a= -1
(asymptotically
stable spiral
point)